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Construction and stability analysis of one-peak symmetric stationary solutions to the Schnakenberg model with heterogeneity

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1 Introduction and Main Results

In this paper, based on a recent work [5], we present our study on the existence and the linear stability of stationary solutions for the following Schnakenberg model:

$$u_t = \varepsilon^2 u_{xx} + d\varepsilon - u + g(x)u^2v, \quad x \in (-1, 1), \quad t > 0, \quad (1)$$

$$\varepsilon v_t = Dv_{xx} + \frac{1}{2} - \frac{c}{\varepsilon}g(x)u^2v, \quad x \in (-1, 1), \quad t > 0, \quad (2)$$

$$u_x(\pm 1) = v_x(\pm 1) = 0, \quad (3)$$

where d and c are positive constants, ε^2 and D are positive diffusion coefficients. $u(x, t)$ and $v(x, t)$ represent the density of two chemical substances. Here, $g(x)$ is a positive function, which represents the reaction speed of the chemical reaction at $x \in (-1, 1)$ and may vary on the location x , for example by the effect of temperature.

Our system (1)-(3) is obtained from the original Schnakenberg model:

$$U_t = D_1 U_{xx} + a - U + g(x)U^2V, \quad x \in (-1, 1), \quad t > 0,$$

$$V_t = D_2 V_{xx} + b - g(x)U^2V, \quad x \in (-1, 1), \quad t > 0,$$

$$U_x(\pm 1) = V_x(\pm 1) = 0$$

by using the spacial scaling: $c = \frac{1}{4b^2}$, $d = ac^{-\frac{1}{2}} = 2ab$, and

$$U = \frac{1}{2b\varepsilon}u, \quad V = 2b\varepsilon v, \quad D_1 = \varepsilon^2, \quad D_2 = \frac{D}{\varepsilon}.$$

Especially, we treat sufficiently small ε and a fixed D , i.e. the ratio of diffusion coefficients $\frac{D}{\varepsilon^2}$ is large (cf. Turing's diffusion-driven instability). Moreover, (2) means that v reacts very rapidly than u in our model.

Inspired by the work of Iron, Wei and Winter [3] which studied in the case $d = 0$ and $g(x) = 1$, the purpose of our study is to investigate the effect of symmetric heterogeneity $g(x)$, namely $g(x) = g(-x)$, on the linear stability of stationary solutions for (1)-(3) rigorously. To state our main results, we prepare some notations. Let w_0 be the unique solution of

$$\begin{aligned} w_0'' - w_0 + w_0^2 &= 0, \quad x \in \mathbb{R}, \\ w_0 &> 0, \quad w_0(0) = \max_{\mathbb{R}} w_0, \quad \lim_{|y| \rightarrow \infty} w_0(y) = 0. \end{aligned}$$

It is known that w_0 is unique and can be written explicitly $w_0(y) = \frac{3}{2}(\cosh \frac{y}{2})^{-2}$. Let w be the unique solution of the following problem:

$$\begin{aligned} w'' - w + g(0)w^2 &= 0, \quad x \in \mathbb{R}, \\ w &> 0, \quad w(0) = \max_{\mathbb{R}} w, \quad \lim_{|y| \rightarrow \infty} w(y) = 0. \end{aligned}$$

Then it is easy to see $w(y) = g(0)^{-1}w_0(y)$. Let χ be a cut-off function:

$$\chi \in C_0^\infty(\mathbb{R}), \quad 0 \leq \chi \leq 1, \quad \chi(x) = \begin{cases} 1, & |x| < \frac{1}{4}, \\ 0, & |x| > \frac{1}{2}. \end{cases}$$

Define symmetric function spaces: for each $a \in (0, \infty)$,

$$\begin{aligned} L_s^2(-a, a) &:= \{u \in L^2(-a, a) \mid u(x) = u(-x)\}, \\ H_s^2(-a, a) &:= \{u \in H^2(-a, a) \mid u(x) = u(-x), u'(\pm a) = 0\}. \end{aligned}$$

Let $I := (-1, 1)$ and $I_\varepsilon := (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$ for $\varepsilon > 0$. We also use the following notation for the rescaling: for a function $u : I \rightarrow \mathbb{R}$, define $\bar{u}(y) := u(\varepsilon y)$ ($y \in I_\varepsilon$).

The steady-state problem for (1)~(3) is the following:

$$0 = \varepsilon^2 u'' + d\varepsilon - u + g(x)u^2v, \quad x \in (-1, 1), \quad (4)$$

$$0 = Dv'' + \frac{1}{2} - \frac{\varepsilon}{D}g(x)u^2v, \quad x \in (-1, 1), \quad (5)$$

$$u'(\pm 1) = v'(\pm 1) = 0. \quad (6)$$

First, we state the existence of a one-peak solution.

Theorem 1 *Fix $D < +\infty$ arbitrarily. Assume that $g(x)$ is positive, Lipschitz continuous and satisfies $g(x) = g(-x)$. Then, there exists a sufficiently small $\varepsilon_1 > 0$ such that, for $0 < \varepsilon < \varepsilon_1$, (4)~(6) admits a symmetric one-peak solution $(u_\varepsilon(x), v_\varepsilon(x)) \in H_s^2(I_\varepsilon) \times H_s^2(I_\varepsilon)$, where $u_\varepsilon(x)$ concentrates at $x = 0$. Moreover, $u_\varepsilon(x)$ takes the following asymptotic form:*

$$u_\varepsilon(x) = w_\varepsilon(x) + \phi_\varepsilon(x), \quad (7)$$

where

$$w_\varepsilon(x) := \frac{1}{\xi_0} w\left(\frac{x}{\varepsilon}\right) \chi(x), \quad \xi_0 := cg(0) \int_{\mathbb{R}} w^2(y) dy$$

and $\overline{\phi_\varepsilon} \in H_s^2(I_\varepsilon)$ such that

$$\|\overline{\phi_\varepsilon}\|_{H^2(I_\varepsilon)} \leq C\sqrt{\varepsilon} \quad (8)$$

holds for some constant $C > 0$ independent of ε . Also, $v_\varepsilon(x)$ satisfies

$$v_\varepsilon(0) = \xi_0 + O(\sqrt{\varepsilon}) \text{ as } \varepsilon \rightarrow 0. \quad (9)$$

Moreover, there exists $v_0 \in H^1(I)$ such that $v_\varepsilon \rightharpoonup v_0$ weakly in $H^1(I)$, where v_0 satisfies

$$\begin{aligned} -Dv_0''(x) &= \frac{1}{2} - \delta_0(x), \quad x \in (-1, 1), \\ v_0(0) &= \xi_0, \quad v_0'(\pm 1) = 0 \end{aligned}$$

and $\delta_0(x)$ is the Dirac's delta function.

Next, we study the linear stability of the solutions $(u_\varepsilon, v_\varepsilon)$ constructed in Theorem 1. We linearize the system (1)~(3) at $(u_\varepsilon, v_\varepsilon)$ and obtain the following eigenvalue problem:

$$\varepsilon^2 \varphi_\varepsilon'' - \varphi_\varepsilon + 2gu_\varepsilon v_\varepsilon \varphi_\varepsilon + gu_\varepsilon^2 \psi_\varepsilon = \lambda_\varepsilon \varphi_\varepsilon, \quad x \in (-1, 1), \quad (10)$$

$$D\psi_\varepsilon'' - \frac{2c}{\varepsilon} gu_\varepsilon v_\varepsilon \varphi_\varepsilon - \frac{c}{\varepsilon} gu_\varepsilon^2 \psi_\varepsilon = \varepsilon \lambda_\varepsilon \psi_\varepsilon, \quad x \in (-1, 1), \quad (11)$$

$$\varphi_\varepsilon'(\pm 1) = \psi_\varepsilon'(\pm 1) = 0,$$

where, λ_ε is an eigenvalue, and $(\varphi_\varepsilon, \psi_\varepsilon) \neq (0, 0)$ is an eigenfunction. We say that the solution $(u_\varepsilon, v_\varepsilon)$ is stable if $\text{Re} \lambda_\varepsilon < 0$ holds for all eigenvalues and unstable if there exists an eigenvalue satisfying $\text{Re} \lambda_\varepsilon > 0$. We have the following result on the stability.

Theorem 2 Fix $D < +\infty$. Let $\varepsilon > 0$ be sufficiently small. Let $(u_\varepsilon, v_\varepsilon)$ be the solution given in Theorem 1. Then, we have the following for large eigenvalues, namely $\lambda_\varepsilon \rightarrow \lambda_0 \neq 0$:

(1) $(u_\varepsilon, v_\varepsilon)$ is stable for any $D < +\infty$, namely $\text{Re}(\lambda_\varepsilon) < 0$ holds.

Furthermore, let $g \in C^3(-1, 1)$. Then, we have the following for small eigenvalues, namely $\lambda_\varepsilon \rightarrow 0$:

(2) If $g''(0) \leq 0$, then $(u_\varepsilon, v_\varepsilon)$ is stable for any $D < +\infty$.

(3) If $g''(0) > 0$, $(u_\varepsilon, v_\varepsilon)$ is stable for $D < D_1$, $(u_\varepsilon, v_\varepsilon)$ is unstable for $D > D_1$, where, $D_1 > 0$ is

$$D_1 := \frac{1}{2c \int_{\mathbb{R}} w_0^2} \cdot \frac{g^2(0)}{g''(0)} = \frac{1}{12c} \cdot \frac{g^2(0)}{g''(0)}.$$

In fact, we have the following asymptotic behavior of λ_ε as $\varepsilon \rightarrow 0$:

$$\lambda_\varepsilon = \varepsilon^2 \frac{\int_{\mathbb{R}} w^3}{\int_{\mathbb{R}} (w')^2} \left(-\frac{g(0)}{6D\xi_0} + \frac{g''(0)}{3} \right) + O(\varepsilon^{\frac{5}{2}}). \quad (12)$$

Remark 1 Note $\int_{\mathbb{R}} w_0^2 = 6$ and

$$\xi_0 = cg(0) \int_{\mathbb{R}} w^2(y) dy = cg(0)^{-1} \int_{\mathbb{R}} w_0^2(y) dy = 6cg(0)^{-1}.$$

Hence, for the case $g''(0) > 0$ the condition $D < D_1$ is equivalent to

$$\left(-\frac{g(0)}{6D\xi_0} + \frac{g''(0)}{3} \right) < 0.$$

Remark 2 Since we are concerned with the existence of unstable eigenvalues, we can assume that $\operatorname{Re}\lambda_\varepsilon \geq -\frac{1}{4}$ for example. We can show that eigenvalues λ_ε are uniformly bounded under the assumption $\operatorname{Re}\lambda_\varepsilon \geq -\frac{1}{4}$. Therefore, we can assume that there exists a λ_0 such that $\lambda_\varepsilon \rightarrow \lambda_0$ as $\varepsilon \rightarrow 0$, taking a subsequence if necessary.

Remark 3 In the case $g(x) = 1$ and $d = 0$, by the result of Iron, Wei, and Winter[3], the one-peak symmetric solution is stable for any $D > 0$. Compared with the case $g(x) = 1$, Theorem 2 reveals the strong influence of the heterogeneity $g(x)$ on the stability of the one-peak symmetric solution. We should mention that a similar destabilization effect of the heterogeneity has been studied for the Gierer-Meinhardt system(see [8]). We note that our results also cover the case $d = 0$. We emphasize that, even in the case $g(x) = 1$, the remainder estimate $O(\varepsilon^{\frac{5}{2}})$ for small eigenvalues is more precise than the result of Iron, Wei and Winter. We also emphasize that for the case $d > 0$ we need to take care of the remainder terms more carefully, compared with the case $d = 0$.

Even for non-symmetric heterogeneity $g(x)$, we can expect similar results. However, we need more computations and left to future works. For the related works with some heterogeneity in other Turing systems, see for example [2], [6], [7], [8] and the references therein. Recently, for a given $N \in 2, N \in \mathbb{N}$ and a given symmetric $\frac{2}{N}$ -periodic function $g(x)$ in the interval $I = (-1, 1)$, one of the authors studied the existence of multi-peak symmetric solutions and its stability in details (see [4]). We also mention that Ao and Liu[1] studied recently another heterogeneity effect on the existence and its stability for the Schnakenberg model with precursors.

2 Outline of the Proof of Theorem 1

2.1 Heuristic explanation of the choice of ξ_0

Before giving the outline of the proof of Theorem 1, we explain briefly why we choose ξ_0 as follows in Theorem 1:

$$\xi_0 := cg(0) \int_{\mathbb{R}} w^2(y) dy.$$

Suppose \overline{u}_ε and \overline{v}_ε are uniformly bounded. Then by the equation for v :

$$-D\overline{v}_\varepsilon'' = \frac{\varepsilon^2}{2} - \varepsilon c \overline{g} \overline{u}_\varepsilon^2 \overline{v}_\varepsilon, \quad (13)$$

we have $|D\overline{v}_\varepsilon''(y)| \leq C\varepsilon$. Since \overline{v}_ε is symmetric, we have $\overline{v}_\varepsilon'(0) = 0$. Therefore, for fixed $R > 0$ we have $|\overline{v}_\varepsilon'(y)| \leq CR\varepsilon$ ($|y| \leq R$). This implies $\overline{v}_\varepsilon(y) \sim C_0$ ($|y| \leq R$) for some positive constant C_0 . On the other hand \overline{u}_ε satisfies

$$-\overline{u}_\varepsilon'' = d\varepsilon - \overline{u}_\varepsilon + \overline{g} \overline{u}_\varepsilon^2 \overline{v}_\varepsilon, \quad y \in I_\varepsilon = (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}).$$

Now we expect $\overline{u}_\varepsilon(y) \sim A_0 w(y) := u_\infty(y)$. Then we have

$$-u_\infty''(y) + u_\infty(y) = g(0)u_\infty(y)^2 C_0, \quad y \in \mathbb{R}.$$

So if we take $w(y)$ to be a solution to $-w'' + w = g(0)w^2$, we must have $A_0 C_0 = 1$. Now integrating (13), we have

$$0 = 1 - c \int_{I_\varepsilon} \overline{g} \overline{u}_\varepsilon^2 \overline{v}_\varepsilon dy.$$

So taking the limit, we would have

$$1 = c \int_{\mathbb{R}} g(0) A_0^2 w(y)^2 C_0 dy.$$

Therefore, since $A_0 C_0 = 1$ we should have

$$A_0 = \frac{1}{cg(0) \int_{\mathbb{R}} w^2 dy}.$$

Thus if we define $\xi_0 := cg(0) \int_{\mathbb{R}} w^2(y) dy$, then we have

$$u_\varepsilon(x) \sim \frac{1}{\xi_0} w\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad v_\varepsilon(0) \sim \xi_0.$$

2.2 Outline of the construction by using the contraction mapping principle.

Let $u = w_\varepsilon + \phi$ with $\overline{\phi}(y) \in B(C_0)$, where

$$w_\varepsilon(x) := \frac{1}{\xi_0} w\left(\frac{x}{\varepsilon}\right) \chi(x), \quad \xi_0 := cg(0) \int_{\mathbb{R}} w^2(y) dy$$

and

$$B(C_0) := \left\{ \overline{\phi} \in H_s^2(I_\varepsilon) \mid \|\overline{\phi}\|_{H^2(I_\varepsilon)} \leq C_0 \sqrt{\varepsilon}, \quad \overline{\phi}'(\pm \frac{1}{\varepsilon}) = 0 \right\}, \quad (14)$$

where the constant C_0 is independent of ε , which will be chosen suitably later. Then, we can find a unique solution $v := T[u] = T[w_\varepsilon + \phi]$ of the second equation (5):

$$-Dv'' + \frac{c}{\varepsilon} g(x) u^2 v = \frac{1}{2}, \quad x \in (-1, 1), \quad v'(\pm 1) = 0.$$

We seek a unique $\phi(x) \in H_s^2(I)$ such that $(u(x), v(x)) = (w_\varepsilon + \phi, T[w_\varepsilon + \phi])$ satisfies the first equation (4). Substituting $u = w_\varepsilon + \phi, v = T[w_\varepsilon + \phi]$ into the first equation (4):

$$-\varepsilon^2 u'' = d\varepsilon - u + g(x)u^2 T[u], \quad x \in (-1, 1), \quad u'(\pm 1) = 0,$$

we have

$$\varepsilon^2 \phi'' - \phi + 2gw_\varepsilon \phi T[w_\varepsilon + \phi] + gw_\varepsilon^2 T[w_\varepsilon + \phi] + \varepsilon^2 w_\varepsilon'' - w_\varepsilon + d\varepsilon + g\phi^2 T[w_\varepsilon + \phi] = 0. \quad (15)$$

Using the Fréchet derivative $R_\varepsilon[\phi] = \langle T'[w_\varepsilon], \phi \rangle$, we have

$$S_\varepsilon[\phi] + gw_\varepsilon^2 T[w_\varepsilon] + \varepsilon^2 w_\varepsilon'' - w_\varepsilon + d\varepsilon + N_1[\phi] = 0,$$

where

$$S_\varepsilon[\phi] := \varepsilon^2 \phi'' - \phi + 2gT[w_\varepsilon]w_\varepsilon \phi + R_\varepsilon[\phi]gw_\varepsilon^2 \quad (16)$$

and $N_1[\phi]$ is the higher order term. Here, in the y -variable, using $\overline{w_\varepsilon''} - \overline{w_\varepsilon} = -\xi_0 g(0)\overline{w_\varepsilon}^2 + O(e^{-\frac{1}{4\varepsilon}})$ we rewrite as follows:

$$\overline{S_\varepsilon[\phi]} + \overline{g} \overline{w_\varepsilon}^2 \overline{T[w_\varepsilon]} - g(0)\xi_0 \overline{w_\varepsilon}^2 + d\varepsilon + O(e^{-\frac{1}{4\varepsilon}}) + \overline{N_1[\phi]} = 0, \quad (17)$$

where

$$\overline{S_\varepsilon[\phi]} := \overline{S_\varepsilon[\phi]} = \overline{\phi''} - \overline{\phi} + 2\overline{g} \overline{T[w_\varepsilon]} \overline{w_\varepsilon} \overline{\phi} + \overline{R_\varepsilon[\phi]} \overline{g} \overline{w_\varepsilon}^2.$$

Now we have the following invertibility of the operator $\overline{S_\varepsilon} : H_s^2(I_\varepsilon) \rightarrow L_s^2(I_\varepsilon)$.

Lemma 1 ([5, Lemma 3.2]) *There exist $\varepsilon_0 > 0$ and $\lambda > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, the following inequality holds:*

$$\left\| \overline{S_\varepsilon[\phi]} \right\|_{L^2(I_\varepsilon)} \geq \lambda \left\| \overline{\phi} \right\|_{H^2(I_\varepsilon)}, \quad \overline{\phi} \in H_s^2(I_\varepsilon). \quad (18)$$

Furthermore, $\text{Ran}(\overline{S_\varepsilon}) = L_s^2(I_\varepsilon)$ holds.

Thus we have

$$\overline{\phi} = -\overline{S_\varepsilon}^{-1} [\overline{T[w_\varepsilon]}] - \overline{S_\varepsilon}^{-1} [\overline{N_1[\phi]}] =: M_\varepsilon[\overline{\phi}], \quad (19)$$

where

$$\overline{T[w_\varepsilon]} := \overline{g} \overline{w_\varepsilon}^2 \overline{T[w_\varepsilon]} - g(0)\xi_0 \overline{w_\varepsilon}^2 + d\varepsilon + O(e^{-\frac{1}{4\varepsilon}}).$$

If we choose $C_0 > 0$ large enough such that

$$\left\| \overline{S_\varepsilon}^{-1} [\overline{T[w_\varepsilon]}] \right\|_{H^2(I_\varepsilon)} \leq \frac{C_0}{2} \sqrt{\varepsilon} \quad (20)$$

we can show that M_ε is a contraction mapping on $B(C_0)$ for small $\varepsilon > 0$. Actually, we can choose C_0 so that $C_0 > \frac{4(C_1+d)}{\lambda}$, where C_1 is the constant will appear in Corollary 1 later. (Note that the constants $\lambda > 0$ and C_1 depend only on $w(x), g(x)$ and the fixed parameters $c > 0, D > 0$.) Thus, there exists a unique $\overline{\phi} \in B(C_0)$ which satisfies the desired equation.

2.3 Basic estimates, including the estimates for $T[w_\varepsilon + \phi]$.

We note the following estimates, which play key roles throughout this work.

Lemma 2 ([5, Lemma 2.8, 2.9]) *Fix $C_0 > 0$. For each $\bar{\phi} \in B(C_0)$, let $\eta(x) \in H^2(-1, 1)$ satisfy*

$$-D\eta'' + \frac{cg(w_\varepsilon + \phi)^2}{\varepsilon}\eta = \frac{h}{\varepsilon}, \quad x \in (-1, 1), \quad \eta'(\pm 1) = 0, \quad (21)$$

where $h(x)$ is a given function on $L^2(-1, 1)$. Then, the following estimates hold:

$$(1) \quad \|\bar{\eta}\|_{L^\infty(I_\varepsilon)} \leq C \|\bar{h}\|_{L^1(I_\varepsilon)}.$$

$$(2) \quad \|\bar{\eta}'\|_{L^2(I_\varepsilon)} \leq C\sqrt{\varepsilon} \|\bar{h}\|_{L^1(I_\varepsilon)}.$$

$$(3) \quad |\bar{\eta}(y)\bar{g}(y) - \bar{\eta}(0)\bar{g}(0)| \leq C\sqrt{\varepsilon|y|} \|\bar{h}\|_{L^1(I_\varepsilon)}, \quad y \in I_\varepsilon.$$

Here, the constant C is independent of ε . Furthermore, if we have a uniform bound $\|\bar{h}\|_{L^1(I_\varepsilon)} \leq M$, then we have

$$\eta(0) = \xi_0 \int_{I_\varepsilon} \bar{h} dy + O(\sqrt{\varepsilon}) \quad \text{as } \varepsilon \rightarrow 0. \quad (22)$$

By using Lemma 2, we can obtain the following estimate which allows the estimate (20).

Corollary 1 *There exists a constant C_1 such that the following estimates hold*

$$\|\overline{T[w_\varepsilon]}\|_{L^\infty(I_\varepsilon)} \leq C_1, \quad |T[w_\varepsilon](0) - \xi_0| \leq C_1\sqrt{\varepsilon},$$

$$\|\bar{g} \bar{w}_\varepsilon^{-2} \overline{T[w_\varepsilon]} - g(0)\xi_0\bar{w}_\varepsilon^{-2}\|_{L^2(I_\varepsilon)} \leq C_1\sqrt{\varepsilon}.$$

(Proof.) By (1), (3) of Lemma 2 and (22) as $\phi = 0$ and $h(x) = \frac{\varepsilon}{2}$, there exists a constant C_1 , independent of ε and C_0 , such that the following estimates hold:

$$\|\overline{T[w_\varepsilon]}\|_{L^\infty(I_\varepsilon)} \leq C_1, \quad |T[w_\varepsilon](0) - \xi_0| \leq C_1\sqrt{\varepsilon},$$

$$|\overline{T[w_\varepsilon]}(y)\bar{g}(y) - \overline{T[w_\varepsilon]}(0)\bar{g}(0)| \leq C_1\sqrt{\varepsilon|y|}, \quad y \in I_\varepsilon.$$

Thus we have

$$\begin{aligned} & \bar{w}_\varepsilon^{-2}(y) |\overline{T[w_\varepsilon]}(y)\bar{g}(y) - \xi_0\bar{g}(0)| \\ & \leq \bar{w}_\varepsilon^{-2}(y) \left(|\overline{T[w_\varepsilon]}(y)\bar{g}(y) - \overline{T[w_\varepsilon]}(0)\bar{g}(0)| + \bar{g}(0)|T[w_\varepsilon](0) - \xi_0| \right) \\ & \leq g(0)C_1\bar{w}_\varepsilon^{-2}(y)(\sqrt{\varepsilon|y|} + \sqrt{\varepsilon}) \leq g(0)\frac{C_1}{\xi_0^2}w^2(y)(\sqrt{\varepsilon|y|} + \sqrt{\varepsilon}), \quad y \in I_\varepsilon. \end{aligned}$$

This implies the desired estimate.

3 Global pointwise estimates for solutions

Since $\overline{u_\varepsilon}(y) = \overline{w_\varepsilon}(y) + \overline{\phi_\varepsilon}(y)$ with $\|\overline{\phi_\varepsilon}\|_{H^2(I_\varepsilon)} \leq C\sqrt{\varepsilon}$, we easily have

$$|\overline{u_\varepsilon}(y)| \leq C\sqrt{\varepsilon} + Ce^{-\frac{|y|}{\sqrt{2}}}, \quad y \in I_\varepsilon$$

by using the Sobolev's embedding theorem. However, this estimate is not enough to treat several error terms in the stability analysis. We need the following pointwise estimates for the solution $(u_\varepsilon, v_\varepsilon)$ in our stability analysis.

Lemma 3 ([5, Proposition 4.2]) *There exists a constant C , which is independent of ε , such that the following estimates hold:*

(i)

$$\|\overline{v_\varepsilon}'\|_{L^\infty(I_\varepsilon)} \leq C\varepsilon,$$

(ii)

$$|\overline{u_\varepsilon}(y)| \leq C(d\varepsilon + e^{-\frac{1}{\sqrt{2}\varepsilon}} + e^{-\frac{|y|}{\sqrt{2}}}), \quad y \in I_\varepsilon,$$

(iii)

$$|\overline{u_\varepsilon}'(y)| \leq C(d^2\varepsilon^2 + e^{-\frac{\sqrt{2}}{\varepsilon}} + e^{-\frac{|y|}{2}}), \quad y \in I_\varepsilon.$$

These estimates can be obtained by using comparison arguments. In particular, $\overline{u_\varepsilon}(y)$ and $\overline{u_\varepsilon}'(y)$ are exponentially small near the boundary of I_ε if $d = 0$. We also have the following uniform bounds:

$$\|\overline{u_\varepsilon}\|_{L^\infty(I_\varepsilon)} \leq C, \quad \|\overline{v_\varepsilon}\|_{L^\infty(I_\varepsilon)} \leq C, \quad \|\overline{u_\varepsilon}\|_{L^2(I_\varepsilon)} \leq C, \quad \|\overline{u_\varepsilon}'\|_{L^2(I_\varepsilon)} \leq C.$$

4 Outline of the Proof of Theorem 2

We may assume that $\|\overline{\varphi_\varepsilon}\|_{H^2(I_\varepsilon)} = 1$. By the extension theorem we have $\|\overline{\varphi_\varepsilon}\|_{H^2(\mathbf{R})} \leq C$. So, there exists a subsequence and $\overline{\varphi} \in H^2(\mathbf{R})$ such that $\overline{\varphi_\varepsilon}$ converges to $\overline{\varphi}$ weakly in $H^2(\mathbf{R})$ and strongly in $C_{loc}^1(\mathbf{R})$.

Lemma 4 (Boundedness of unstable eigenvalues, [5, Proposition 4.2]) *Assume $\operatorname{Re}(\lambda_\varepsilon) \geq -\frac{1}{4}$. Then, we have the following:*

(1) $\overline{\varphi} \neq 0$.

(2) *There exists a constant C , independent of ε , such that $|\lambda_\varepsilon| \leq C$.*

By this Lemma, we may assume $\lambda_\varepsilon \rightarrow \lambda_0$ for some constant λ_0 . We consider two cases:

(a) **large eigenvalue:** i.e. $\lambda_\varepsilon \rightarrow \lambda_0 \neq 0$.

(b) **small eigenvalue:** i.e. $\lambda_\varepsilon \rightarrow 0$.

4.1 Stability analysis for large eigenvalues

Lemma 5 Assume $\lambda_\varepsilon \rightarrow \lambda_0 \neq 0$. Then, we have

$$\overline{\varphi}''(y) - \overline{\varphi}(y) + 2w_0(y)\overline{\varphi}(y) - 2\frac{\int w_0 \overline{\varphi} dz}{\int w_0^2 dz} w_0(y)^2 = \lambda_0 \overline{\varphi}(y).$$

Then, by the well-known lemma of Wei and Winter (see Lemma 2.2 in [3], or [8]) for nonlocal eigenvalue problem above, we can conclude $\text{Re}\lambda_0 < 0$. So for sufficiently small $\varepsilon > 0$ we have $\text{Re}\lambda_\varepsilon < 0$, namely λ_ε is a stable eigenvalue.

(Sketch of the proof of Lemma 5.) From the equation (11) for ψ_ε , we can show $\|\psi_\varepsilon\|_{H^1(I)} \leq C$ and apply Lemma 2 to obtain

$$\begin{aligned} \overline{\psi}_\varepsilon(0) &= \psi_\varepsilon(0) = \xi_0 \int_{I_\varepsilon} (-2c\overline{g u_\varepsilon v_\varepsilon \varphi_\varepsilon} - \varepsilon^2 \lambda_\varepsilon \overline{\psi_\varepsilon}) dy + O(\sqrt{\varepsilon}) \\ \rightarrow \psi(0) &= -2cg(0)\xi_0 \int_{\mathbb{R}} w \overline{\varphi} dy = -2\xi_0^2 \frac{\int_{\mathbb{R}} w \overline{\varphi} dy}{\int_{\mathbb{R}} w^2 dy} \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (23)$$

Here we used $\xi_0 = cg(0) \int_{\mathbb{R}} w^2 dy$. On the other hand, from the equation (10), we have

$$\overline{\varphi}_\varepsilon'' - \overline{\varphi}_\varepsilon + 2\overline{g u_\varepsilon v_\varepsilon \varphi_\varepsilon} + \overline{g u_\varepsilon^2 \psi_\varepsilon} = \lambda_\varepsilon \overline{\varphi}_\varepsilon \text{ on } I_\varepsilon.$$

Then, for any $\zeta \in C_0^\infty(\mathbb{R})$, we have

$$\int_{I_\varepsilon} \left(\overline{\varphi}_\varepsilon'' - \overline{\varphi}_\varepsilon + 2\overline{g u_\varepsilon v_\varepsilon \varphi_\varepsilon} + \overline{g u_\varepsilon^2 \psi_\varepsilon} \right) \zeta(y) dy = \int_{\mathbb{R}} \lambda_\varepsilon \overline{\varphi}_\varepsilon \zeta(y) dy.$$

Taking $\varepsilon \rightarrow 0$ and using Lemma 2, we obtain

$$\int_{\mathbb{R}} \left(\overline{\varphi}''(y) - \overline{\varphi}(y) + 2g(0)w(y)\overline{\varphi}(y) + \frac{g(0)}{\xi_0^2} w^2(y)\psi(0) \right) \zeta(y) dy = \lambda_0 \int_{\mathbb{R}} \overline{\varphi}(y)\zeta(y) dy. \quad (24)$$

Since $w(y) = g(0)^{-1}w_0(y)$, (23) and (24) yield Lemma 5.

4.2 Stability analysis for small eigenvalues

We have the following key precise asymptotic for small eigenvalues λ_ε which yields the proof of Theorem 2.

Proposition 1 Assume $\lambda_\varepsilon \rightarrow 0$. Then, as $\varepsilon \rightarrow 0$, the asymptotic form of λ_ε is given as follows:

$$\lambda_\varepsilon = \varepsilon^2 \frac{\int_{\mathbb{R}} w^3}{\int_{\mathbb{R}} (w')^2} \left(-\frac{g(0)}{6D\xi_0} + \frac{g''(0)}{3} \right) + O(\varepsilon^{\frac{5}{2}}). \quad (25)$$

(Sketch of the proof of Proposition 1.) We denote $u_{\epsilon,1}(x) = u_\epsilon(x)\chi(x)$. To show Proposition 1, let us decompose

$$\varphi_\epsilon(x) = \varepsilon a_\epsilon u'_{\epsilon,1}(x) + \varphi_\epsilon^\perp(x), \quad (26)$$

where a_ϵ is some complex number and φ_ϵ^\perp satisfies

$$\overline{\varphi_\epsilon^\perp} \perp \mathcal{K}_\epsilon \text{ in } L^2(I_\epsilon), \quad \mathcal{K}_\epsilon := \text{span}\{\overline{u_{\epsilon,1}}'\} \subset H^2(I_\epsilon).$$

In y -variable, we have

$$\overline{\varphi_\epsilon}(y) = a_\epsilon \overline{u_{\epsilon,1}}'(y) + \overline{\varphi_\epsilon^\perp}(y).$$

Similarly, we decompose

$$\psi_\epsilon(x) = \varepsilon a_\epsilon \psi_{\epsilon,1}(x) + \psi_\epsilon^\perp(x). \quad (27)$$

Here, $\psi_{\epsilon,1}$ is a unique solution of

$$D\psi_{\epsilon,1}'' - \frac{c}{\varepsilon}g(x)u_\epsilon^2\psi_{\epsilon,1} - \frac{2c}{\varepsilon}g(x)v_\epsilon u'_{\epsilon,1} = \varepsilon\lambda_\epsilon\psi_{\epsilon,1}, \quad x \in (-1, 1), \quad \psi'_{\epsilon,1}(\pm 1) = 0, \quad (28)$$

and ψ_ϵ^\perp is defined by $\psi_\epsilon^\perp := \psi_\epsilon - \varepsilon a_\epsilon \psi_{\epsilon,1}$. We have the following formula for λ_ϵ :

Lemma 6 ([5, Lemma 6.1])

$$J_1 + J_2 + J_3 + J_4 + O(|a_\epsilon|\varepsilon^3) = \lambda_\epsilon a_\epsilon \xi_0^{-2} \int_{\mathbb{R}} (w')^2 dy + O(\sqrt{\varepsilon}\lambda_\epsilon|a_\epsilon|). \quad (29)$$

where J_i ($i = 1, 2, 3, 4$) are defined as follows:

$$\begin{aligned} J_1 &:= a_\epsilon \int_{I_\epsilon} (\varepsilon \overline{\psi_{\epsilon,1}} - \overline{v_\epsilon'}) \overline{u_{\epsilon,1}}^2 \overline{u_{\epsilon,1}}' dy - a_\epsilon \int_{I_\epsilon} \overline{g'v_\epsilon} \overline{u_{\epsilon,1}}^2 \overline{u_{\epsilon,1}}' dy, \\ J_2 &:= - \int_{I_\epsilon} (\overline{g'v_\epsilon} + \overline{g} \overline{v_\epsilon'}) \overline{u_{\epsilon,1}}^2 \overline{\varphi_\epsilon^\perp} dy, \\ J_3 &:= \int_{I_\epsilon} \overline{g} \overline{u_\epsilon}^2 \overline{\psi_\epsilon^\perp} \overline{u_{\epsilon,1}}' dy, \quad J_4 := \int_{I_\epsilon} \overline{r_\epsilon} \overline{\varphi_\epsilon^\perp} dy. \end{aligned}$$

Here, $\overline{r_\epsilon}$ is a function satisfying $\overline{r_\epsilon}(y) = 0$ on $|y| \leq \frac{1}{4\varepsilon}$ and $\overline{r_\epsilon}(y) = O(\varepsilon^2)$ on $\frac{1}{4\varepsilon} \leq |y| \leq \frac{1}{\varepsilon}$.

This is obtained by multiplying $\overline{u_{\epsilon,1}}'$ and integration by parts. Among them, J_1 is the leading term to determine the precise asymptotic for λ_ϵ . The following Proposition 2 is important and decide the asymptotic behavior of J_1 .

Proposition 2 ([5, Proposition 6.2]) *The following estimates hold:*

$$(1) \quad \left\| (\varepsilon \overline{\psi_{\epsilon,1}} - \overline{v_\epsilon'}) \overline{g} \overline{u_\epsilon}^2 \right\|_{L^1(I_\epsilon)} \leq C\varepsilon^2.$$

$$(2) \quad |\varepsilon \overline{\psi_{\epsilon,1}}(y) - \overline{v_\epsilon'}(y)| \leq C\varepsilon^2|y|.$$

$$(3) \quad \varepsilon \overline{\psi_{\varepsilon,1}}(y) - \overline{v_\varepsilon'}(y) = \varepsilon^2 y \cdot \frac{cg(0)\xi_0^{-1}}{2D} \int_{\mathbb{R}} w^2 dt + O(\varepsilon^{\frac{5}{2}}|y|).$$

Here, the constant C is independent of $\varepsilon > 0$.

These are obtained by the representation of $\overline{v_\varepsilon'}(y)$ and $\overline{\psi_{\varepsilon,1}}(y)$ by using Dirichlet and Neumann Green functions, respectively. By Proposition 2, we have the following.

Proposition 3 *It holds that*

$$J_1 = a_\varepsilon \varepsilon^2 \left(-\frac{g(0) \int_{\mathbb{R}} w^3}{6D\xi_0^3} + \frac{g''(0) \int_{\mathbb{R}} w^3}{3\xi_0^2} \right) + O(|a_\varepsilon| \varepsilon^{\frac{5}{2}}),$$

where the constant C is independent of $\varepsilon > 0$.

(Sketch of the proof of Proposition 3.) By (3) of Proposition 2, we have

$$\begin{aligned} & a_\varepsilon \int_{I_\varepsilon} (\varepsilon \overline{\psi_{\varepsilon,1}}(y) - \overline{v_\varepsilon'}(y)) \overline{u_{\varepsilon,1}}^2 \overline{u_{\varepsilon,1}'} dy \\ &= a_\varepsilon \varepsilon^2 \frac{cg(0)\xi_0^{-1}}{2D} \left(\int_{\mathbb{R}} w^2 dx \right) \int_{\mathbb{R}} y \overline{u_{\varepsilon,1}}^2 \overline{u_{\varepsilon,1}'} dy + O(|a_\varepsilon| \varepsilon^{\frac{5}{2}}) \\ &= -a_\varepsilon \varepsilon^2 \frac{cg(0)^2 \int_{\mathbb{R}} w^2 dx}{6D\xi_0^4} \int_{\mathbb{R}} w^3 dx + O(|a_\varepsilon| \varepsilon^{\frac{5}{2}}) = -a_\varepsilon \varepsilon^2 \frac{g(0)}{6D\xi_0^3} \int_{\mathbb{R}} w^3 dx + O(|a_\varepsilon| \varepsilon^{\frac{5}{2}}). \end{aligned}$$

Here, we used

$$cg(0) \int_{\mathbb{R}} w(y)^2 dy = \xi_0, \quad \int_{\mathbb{R}} y w(y)^2 w'(y) dy = -\frac{1}{3} \int_{\mathbb{R}} w(y)^3 dy.$$

Since by using $\overline{g'}(y) = \varepsilon^2 y g''(0) + O(\varepsilon^3 |y|^2)$ we also have

$$\int_{I_\varepsilon} \overline{g' v_\varepsilon u_{\varepsilon,1}^2 u_{\varepsilon,1}'} dy = -\varepsilon^2 \frac{g''(0)\xi_0^{-2}}{3} \int_{\mathbb{R}} w^3 dx + O(\varepsilon^{\frac{5}{2}}), \quad (30)$$

we can conclude that

$$\begin{aligned} J_1 &= a_\varepsilon \int_{I_\varepsilon} (\varepsilon \overline{\psi_{\varepsilon,1}} - \overline{v_\varepsilon'}) \overline{u_{\varepsilon,1}}^2 \overline{u_{\varepsilon,1}'} dy - a_\varepsilon \int_{I_\varepsilon} \overline{g' v_\varepsilon} \overline{u_{\varepsilon,1}}^2 \overline{u_{\varepsilon,1}'} dy \\ &= a_\varepsilon \varepsilon^2 \left(-\frac{g(0) \int_{\mathbb{R}} w^3}{6D\xi_0^3} + \frac{g''(0) \int_{\mathbb{R}} w^3}{3\xi_0^2} \right) + O(|a_\varepsilon| \varepsilon^{\frac{5}{2}}). \end{aligned}$$

To estimate the remainder terms J_2, J_3 , and J_4 , we need the following several estimates.

Lemma 7 ([5, Lemma 6.6]) *For ψ_ε^\perp , it holds that:*

$$(1) \quad \|\overline{\psi_\varepsilon^\perp}\|_{L^\infty(I_\varepsilon)} \leq C \|\overline{\varphi_\varepsilon^\perp}\|_{L^2(I_\varepsilon)}.$$

$$(2) \quad \|\overline{\psi_\varepsilon^\perp'}\|_{L^2(I_\varepsilon)} \leq C \sqrt{\varepsilon} \|\overline{\varphi_\varepsilon^\perp}\|_{L^2(I_\varepsilon)}.$$

Basically, these estimates can be obtained by applying Lemma 2 for ψ_ε^\perp . Lemma 7 implies the following estimates.

Lemma 8 ([5, Lemma 6.8]) *For J_i ($i = 2, 3, 4$), it holds that:*

$$(1) |J_2| \leq C\varepsilon \|\overline{\varphi_\varepsilon^\perp}\|_{L^2(I_\varepsilon)}.$$

$$(2) |J_3| \leq C\sqrt{\varepsilon} \|\overline{\varphi_\varepsilon^\perp}\|_{L^2(I_\varepsilon)}.$$

$$(3) |J_4| \leq C\varepsilon^{\frac{3}{2}} \|\overline{\varphi_\varepsilon^\perp}\|_{L^2(I_\varepsilon)}.$$

Here, the constant C is independent of $\varepsilon > 0$.

By using Proposition 2 again and the invertibility of some operator \tilde{L}_ε , which is close to the operator S_ε , we can show the following estimates.

Lemma 9 ([5, Lemma 6.9]) *The following hold:*

$$(1) |a_\varepsilon| \neq 0.$$

$$(2) \|\overline{\varphi_\varepsilon^\perp}\|_{L^2} \leq C|a_\varepsilon|\varepsilon^{\frac{3}{2}}.$$

Combining these estimates, we arrive at the remainder estimates for J_2, J_3 and J_4 .

Lemma 10 $J_2 = O(|a_\varepsilon|\varepsilon^{\frac{5}{2}})$, $J_3 = O(|a_\varepsilon|\varepsilon^{\frac{5}{2}})$, $J_4 = O(|a_\varepsilon|\varepsilon^3)$.

Estimates for J_2 and J_4 follow directly from Lemma 8 and 9. However, for J_3 , Lemma 8 and 9 yield just $J_3 = (|a_\varepsilon|\varepsilon^2)$, which is not enough. Actually, we need the following refined estimate to get the correct estimate for J_3 .

$$\overline{\psi_\varepsilon^\perp}(y) - \overline{\psi_\varepsilon^\perp}(0) = O(|a_\varepsilon|\varepsilon^{\frac{5}{2}}|y|). \quad (31)$$

This is obtained by the representation of $\overline{\psi_\varepsilon^\perp}(y)$ by using the Neumann Green function. Now, by using Proposition 3, Lemma 10 and Lemma 6, we can complete the proof of Proposition 1.

5 Further Remarks

We give two remarks.

Remark 4 Assume $g(x)$ is Lipschitz continuous and $g \in C^3((-1, 0])$ and $g \in C^3([0, 1))$, respectively. Let $g'(+0) := \lim_{x>0, x \rightarrow 0} g'(x)$ and $g'(-0) := \lim_{x<0, x \rightarrow 0} g'(x) = -g'(+0)$ by the symmetry. When $g'(+0) \neq g'(-0)$, the stability of the solution is determined by

the sign of $g'(+0)$. First, note that using $\bar{g}'(y) = \varepsilon g'(+0) + \varepsilon^2 y g''(+0) + O(\varepsilon^3 |y|^2)$ for $y > 0$ and $\bar{g}'\overline{v_\varepsilon u_\varepsilon^2 u_\varepsilon}'$ is an even function, we can compute

$$\begin{aligned} \int_{I_\varepsilon} \bar{g}'\overline{v_\varepsilon u_\varepsilon^2 u_\varepsilon}' dy &= 2 \int_0^{\frac{1}{\varepsilon}} \bar{g}'\overline{v_\varepsilon u_\varepsilon^2 u_\varepsilon}' dy \\ &= \varepsilon g'(+0) \xi_0^{-2} \int_0^{+\infty} w(y)^2 w'(y) dy + O(\varepsilon^{\frac{3}{2}}) = -\varepsilon \frac{g'(+0)w(0)^3}{3\xi_0^2} + O(\varepsilon^{\frac{3}{2}}). \end{aligned}$$

Thus, in the computation of the small eigenvalue λ_ε , the leading term of J_1 become as follows:

$$J_1 = a_\varepsilon \frac{\varepsilon}{3\xi_0^2} g'(+0)w(0)^3 + O(|a_\varepsilon|\varepsilon^{\frac{3}{2}}).$$

Compare with (30) and Proposition 3 for the case $g \in C^3(-1, 1)$. This implies

$$\lambda_\varepsilon = \varepsilon \frac{g'(+0)}{3 \int_{\mathbb{R}} w'(x)^2 dx} (w(0))^3 + O(\varepsilon^{\frac{3}{2}}).$$

Therefore, the solution is unstable if $g'(+0) > 0$ and stable if $g'(+0) < 0$.

Remark 5 (Boundary peak solution and its stability) For a given Lipschitz continuous positive function $g(x)$, we can construct a boundary peak solution $(u_\varepsilon, v_\varepsilon)$ on the interval $I := (-1, 1)$. Because, consider an extension of $g(x)$ on the interval $\tilde{I} := (-1, 3)$, which is symmetric with respect to $x = 1$. We denote it by $\tilde{g}(x)$. For this function \tilde{g} , we can construct solution $(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon)$ to the corresponding Schnakenberg system on the interval \tilde{I} , which is symmetric with respect to $x = 1$. Restricting this solution on the original interval I , we obtain a boundary peak solution $(u_\varepsilon, v_\varepsilon)$. For the stability of this boundary peak solution, let us consider the linearized eigenvalue problem on I . We denote by λ_ε and $(\varphi_\varepsilon, \psi_\varepsilon)$ the eigenvalue and the associated eigenfunctions, respectively. Now, extending the eigenfunction $(\varphi_\varepsilon, \psi_\varepsilon)$ on the interval $\tilde{I} = (-1, 3)$ to be symmetric with respect to $x = 1$. Then by the Neumann boundary condition at $x = 1$, this extended function $(\tilde{\varphi}_\varepsilon, \tilde{\psi}_\varepsilon)$ is an eigenfunction associated with the eigenvalue λ_ε on the interval \tilde{I} . Then, we can apply our theorem to study the stability of the boundary peak solution $(u_\varepsilon, v_\varepsilon)$. Namely, assuming $\tilde{g}(x)$ is C^3 function, $\tilde{g}''(1)$ determine the stability of the boundary peak solution constructed in this way.

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